The Golden Section and Continued Fractions

The rectangle on the right has dimensions 16 x 45 units.

1. Starting from the left-hand edge, draw two vertical lines in the rectangle so that you have two squares and then a rectangle left over. Satisfy yourself that this is an equivalent mathematical statement:

$$\frac{45}{16}=\frac{16+16+13}{16}=2+\frac{13}{16}$$

2. On squared paper, draw a rectangle which is 13 units by 16 units (so equivalent to the rectangle left after drawing the squares above). Draw a line so that you have a square and a rectangle. Again satisfy yourself that an equivalent mathematical statement is:

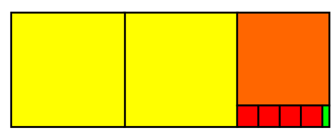
$$\frac{16}{13}=\frac{13+3}{13}=1+\frac{3}{13}$$

3. Repeat for a rectangle which is 3 units by 13 units, and write down an equivalent mathematical statement.

This final stage should give $\frac{13}{3}=\frac{3+3+3+3+1}{3}=4+\frac{1}{3}$

We can put all these stages together to create a *continued fraction*, which consists of whole numbers and *unit fractions* (these are fractions with numerator equal to 1). To follow this, you just need to remember that, eg. $\frac{13}{16}=\frac{1}{16/13}$ because they are *reciprocal* to each other.

$$\frac{45}{16}=2+\frac{13}{16}=2+\frac{1}{16/13}=2+\frac{1}{1+\frac{3}{13}}=2+\frac{1}{1+\frac{1}{13/3}}=2+\frac{1}{1+\frac{1}{4+\frac{1}{3}}}$$



You can check this is correct by evaluating $2+1/(1+1/(4+1/3))$ on your calculator.

You can continue to draw rectangles and squares if you wish, or just use the numbers:

- Step1: Divide the numerator (top) by the denominator (bottom) (omit this stage if the numerator is smaller than the denominator).
- Step 2: Write down the whole number obtained and the remainder as a fraction of the denominator.
- Step 3: Change this fraction to its reciprocal (expressed as 1/another fraction).
- Step 4: Repeat Steps 1 to 3 until either the numerator or denominator is equal to 1.

4. Express $\frac{24}{7}$ as a continued fraction, and check that you are correct with your calculator.

You will have noticed that it's a bit of a pain to write a continued fraction as we have been doing so far! There is an alternative. The continued fraction for 45/16 can be written [2; 1, 4, 3] – check you can see where the figures come from by comparing it with what is written above. The continued fraction for 24/7 in this form is [3; 2, 3] – does this agree with what you got?

Here's a quick way to check:

[3; 2, 3] = [3; 2 + 1/3] = [3; 7/3] = 3 + 3/7 = 24/7

5. Find the improper fractions (ie. a single fraction, with numerator greater than denominator if necessary) for these continued fractions:

[1; 2, 3, 2] and [2; 1, 4, 5]

But what about if the number you start with is a decimal fraction? We use the same process of taking out whole numbers, then looking at the reciprocal of the remainder. So for 2.75:

- 2.75 = [2; …]
- The remainder is .75 so its reciprocal is 1/.75 = 1.3333333…, so we have [2; 1, …]
- The remainder is .3333… so its reciprocal is 1/.3333…, giving [2; 1, 3]

This is easy to do on a calculator, since you just need to subtract the whole number part at each stage, and then use the reciprocal or x^{-1} key. Stop when you no longer have a remainder.

6. Find the continued fraction for 3.15

We can also find continued fractions for the roots of quadratic equations, and the point we have been working up to is to do this for the equation we used earlier to find a value for *phi*.

Suppose we have the equation $x^2 - 3x - 2 = 0$. This can be written:

$$x^2 = 3x + 2$$

$$\text{or } x = 3 + \frac{2}{x} \text{ if we divide through by } x$$

So x can be replaced by $3 + \frac{2}{x}$:

$$x = 3 + \frac{2}{x} = 3 + \frac{2}{3 + \frac{2}{x}} = 3 + \frac{2}{3 + \frac{2}{3 + \frac{2}{x}}} = \ldots$$

giving us the repeated fraction [3; 2, 2, 2, …]

7. Starting with any version of the quadratic equation we found for *phi* earlier today, find its repeated fraction. Does what you find surprise you?

8. Try finding the continued fraction for the decimal value of *phi*. Do you get the same continued fraction?