Life-saving maths:
How does vaccination work?

Teacher Notes

Navigating the Pack

This pack is intended to introduce 13-15 year-old students (Years 9 - 11 in the UK) to the idea of investigating vaccination using maths. Vaccination can be a very emotive issue, with people of strong opinion on both sides of any argument about it. The purpose of this pack is to help students understand the basis for the scientific arguments so that they can form rational opinions, based on evidence.

Below is the list as on the webpage for this pack. This suggests an order in which the resources might be used. Any resources can, of course, be used independently of the rest, but students will gain more from the activities if they have seen the preceding video clips first.

Some of the resources in this pack are also included in the pack *Epidemics: Modelling with mathematics.*

* indicates a resource suitable for almost all students *** indicates more challenging resources

Type of Resource	Resource Name	Difficulty	Notes
	Teacher notes and worksheet answers		Start here!
	How does vaccination work: Introduction	*	Introductory video clip (3 min 38 secs) - Dr Andrew Conlan
	The measles graph	*	Follow-up discussion on patterns in the measles graph
	Measles and vaccination	**	Video clip (5 mins 53 secs) - Dr Andrew Conlan
	Standing Disease	*	Activity: a simple model of epidemic spread
	Graphing Change	**	Worksheet: three contexts (including Standing Disease) comparing different scale representations of the data, and three different representations of the measles data discussed in the first two video clips.
	Counter Plague	*	Activity: varying the effect of a contact

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Type of Resource	Resource Name	Difficulty	Notes
	Counter Plague: demonstration	*	Video clip (3 min 47 secs) - you could omit this if the activity instructions are clear without it, or use it as a teacher resource.
	26-Card Disease	**	Activity: Refining the model to take account of immunity (natural or induced through vaccination)
	26-Card Disease: demonstration	**	Video clip (7 mins 37 secs) - you could omit this if the activity instructions are clear without it, or use it as a teacher resource.
	Reproductive Ratio	**	Video clip (3 mins 56 secs) - Dr Andrew Conlan
	Calculating R_0: Counter Plague	**	Presentation: how do we calculate the reproductive ratio for Counter Plague?
	Immunisation and Reproductive Ratio	**	Video clip (3 mins 41 secs) - Dr Julia Gog
	How many people do we need to vaccinate?	**	Presentation: how do we calculate the proportion of the population which needs to be vaccinated to produce herd immunity?
	e-Counter Plague	**	Activity: a set of 4 online simulations to help students investigate the effect of vaccination.
	Who should be immunised?	*	Video clip (2 mins 10 secs) - Dr Julia Gog
	Network Disease	*	Activity: identifying the right people to vaccinate.
	Investigating Networks	***	Worksheet: identifying the key person in a network.

Standing Disease

This is a very simple model for an epidemic, in which one student chooses two others, they each choose two others, and so on. The number of people 'infected' at each stage is a power of 2 - 1, 2, 4, 8, ...

In general, it would take 4 to 6 steps for a whole class to be infected - $2^4 = 16$; $2^5 = 32$; $2^6 = 64$.

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If 3 people are infected at each stage, the sequence is the powers of 3 - 1, 3, 9, 27, ..., and for *n* people, it is the powers of *n* - $n, n^2, n^3, \ldots$

The world population is about 6.9 billion (2011). If 2 people are infected at each step, then it only takes 33 steps (counting the first infection as step 0) for the entire world population to be infected. If the infection rate is 3, then it only takes 21 steps.

Clearly real epidemics don't follow this model after the first few steps, where exponential increase might well be seen. Reasons include: this model does not take into account people who recover and are then immune, or people who were immune in the first place, or isolated communities.

You could follow this by focusing on graphing data from Standing Disease and measles, comparing these with other natural phenomena, such as the energy liberated in an earthquake and the increase in the population of the world. Alternatively, you could look at other models of epidemic spread, such as Counter Plague and 26-Card Disease.

Graphing Change

The three pairs of graphs provide two ways of displaying data, where there is rapid increase or decrease.

Standing Disease: The graphs show how the number of people affected increases step by step. The population of the world is approaching 7 billion, and starting from 1 person (step 0) and doubling at every step, it only takes 33 steps for this figure to be exceeded, since $2^{33} = 8{,}589{,}934{,}592$. The vertical axis on the left-hand graph is linear, with 1 billion added at each scale point. The vertical axis on the right-hand graph is logarithmic, meaning that it increases by a power of 10 with each scale point.

Earthquakes: The graphs show how the energy released by an earthquake in MJ corresponds to its rating on the Richter scale. As is well known, the Richter scale is a logarithmic scale, meaning that an earthquake magnitude 6 is 10 times more powerful than one of magnitude 5. The vertical scale on the left-hand graph is also logarithmic, increasing by a factor of 100 MJ for each scale point, whereas the vertical scale on the right-hand graph is linear, adding 10^{15} MJ at each scale point.

World population: The graphs show how the population of the world is estimated to have increased between 1700 and 2000. The vertical scale on the left-hand graph is linear with an additional 1 billion people at each scale point. The vertical scale on the right-hand graph is logarithmic, increasing by a factor of 10 at each scale point.

Measles: There are three representations of the measles data discussed by Dr Andrew Conlan in the video clip *Measles and Vaccination*. The first shows the data for the UK between 1940 and 2008. The second shows the data between 1990 and 2008. In the first graph, it appears that there are virtually no cases in this period, because of the scale on the vertical axis. These graphs both have linear scales on the vertical axes, but the scale on the first graph is in hundreds of thousands of people, whereas on the second it is in tens of thousands of people. Clearly the scale affects the detail that can be seen. The third graph shows the same data as the second graph, but with a logarithmic scale on the vertical axis. Again this affects the detail that can be seen.

Students should make their own observations about what they think the advantages and disadvantages of each representation are. It is important to emphasise that these are all perfectly valid ways of displaying data, and that logarithmic scales are frequently used for data which is subject to rapid increase or decrease, not least because this may result in a straight line or approximately straight line graph.

Counter Plague

This model uses a die to introduce variability into the model. The number of people infected at each step in an epidemic is determined by a die, which means we can have an infection rate which isn't a simple whole number.

There is an additional video on the activity itself, which could be used in the classroom, or as part of teacher preparation.

Two scenarios are suggested for investigation:

Number on die	Number of people infected
1	0
2	0
3	1
4	1
5	2
6	3

Number on die	Number of people infected
1	0
2	0
3	1
4	1
5	1
6	2

Given that each of the outcomes on the die are equally likely, the expected rate of infection for the left-hand table is:

$$\frac{1}{6}\times 0+\frac{1}{6}\times 0+\frac{1}{6}\times 1+\frac{1}{6}\times 1+\frac{1}{6}\times 2+\frac{1}{6}\times 3=\frac{7}{6}$$

As this is > 1, then on average, we should expect these epidemics to take off, so that students need to use all the counters available to them.

For the right-hand table, however, the expected rate of infection is:

$$\frac{1}{6}\times 0+\frac{1}{6}\times 0+\frac{1}{6}\times 1+\frac{1}{6}\times 1+\frac{1}{6}\times 1+\frac{1}{6}\times 2=\frac{5}{6}$$

This is < 1, so we should expect on average that epidemics will die out quite quickly.

Graphing the results of several 'epidemics' should help students to see the difference. A bar chart is to be preferred to a line graph, since the data is discrete.

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This model is better than the Standing Disease, because it allows variability in the rate of infection. Since some of the results on the die mean that no one is infected, this can be interpreted either as a sick person choosing to isolate themselves, or as contact with someone who is already immune.

26-card Disease

You may wish to use the video that accompanies this activity as preparation, or for students to view.

This model makes explicit the effect of people recovering and can also be used to model what happens when people are immunised, if some black cards are included in the population right from the start.

Keeping the population at 26 is a way of simulating what happens in a community of a given size, rather than looking at what could happen if contacts outside the community are also considered. Even with two potential infections per infected person, the epidemic rarely affects everyone. This provides a reasonable model of what would happen in, say, a city where most people live and work in the city with few outside contacts.

The probability that no one is infected at Step 1 is 0. The first red card is replaced by one black card, but two cards are placed on the table at Step 1, so at least one of them must be red. To see what might happen subsequently, we need to find a way to record the possible outcomes.

Here are the first three steps (bearing in mind that a black card is not left on the table, but replaced in the population pack before moving on to the next step):

	B available	Possible outcomes
Step 0	0	R
Step 1	1	RR R(B)

Outcome at Step 1	B available	Possible outcomes at Step 2
RR	3	RRRR RRR(B) RR(BB) R(BBB)
R	2	RR R(B) (BB)

The probability that the epidemic terminates at Step 2 is given by:

$$
\begin{aligned}
& p(B \text{ at Step1}) \times p(BB \text{ at Step2}) \\
& = \frac{1}{26} \times \left(\frac{2}{26} \times \frac{1}{25} \right) \\
& = \frac{2}{16{,}900} \approx 0.000118
\end{aligned}
$$

e-Counter Plague

The presentation is designed to help students understand the animations in the e-Counter Plague set.

The screen shot shows the simplest version.

Students can set (bottom left grid):

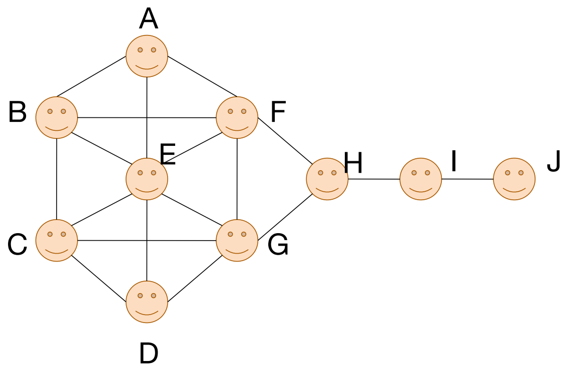
- number of susceptible green smileys
- number of initial infected red smileys
- number of initial immune blue smileys

The die can be set (top left grid) so that infections occur randomly (see Counter Plague above for details). An expected rate of infection of less than 1 should mean that on average, epidemics take off, and all the green smileys (susceptible to the disease) are gradually infected (red) then recover (blue). An expected rate of infection of less than 1 should mean that on average, epidemics do not take off, and some green smileys remain when the epidemic terminates. An expected rate of infection equal to 1 is unpredictable.

Once students have grasped the simple version, they can then look at two-class versions, which show what happens when an infection passes from one classroom to another, and perhaps back again.

The model also applies to any two communities where there are contacts between them.

Investigating Networks



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There is no single right answer here. Students should be encouraged to find their own methods of deciding on who the key person in this network is.

Methods which are used to investigate social networks include:

(x) degree centrality - who has the most direct links to other people (to find this, I worked out the number of direct links for each person, their *degree* in the network)

(y) closeness centrality - who is closest to everyone else, who can access everyone else in the network most easily (to find this, I tabulated the length of the shortest path between each pair of people)

(z) betweenness centrality - who is on most shortest paths between people, so is most important in linking other people to each other (to find this, I recorded the individuals on the shortest path between each pair of people)

Degree centrality:

Node	A	B	C	D	E	F	G	H	I	J
Degree	3	4	4	3	6	5	5	3	2	1

E, at degree 6, has the most direct links with other nodes.

Closeness centrality:

How many steps are there in the shortest path between each pair of nodes?

	A	B	C	D	E	F	G	H	I	J	Sum
A	-	1	2	2	1	1	2	2	3	4	18
B	1	-	1	2	1	1	2	2	3	4	17
C	2	1	-	1	1	2	1	2	3	4	17
D	2	2	1	-	1	2	1	2	3	4	18
E	1	1	1	1	-	1	1	2	3	4	15
F	1	1	2	2	1	-	1	1	2	3	14
G	2	2	1	1	1	1	-	1	2	3	14
H	2	2	2	2	2	1	1	-	1	2	15
I	3	3	3	3	3	2	2	1	-	1	21
J	4	4	4	4	4	3	3	2	1	-	29

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The two nodes with the shortest total number of steps in their shortest paths to all other nodes are F and G at 14 each, so these are closest to all the other nodes.

Betweenness centrality:

For this, I tabulated the intermediate nodes on the shortest path(s) between each pair of nodes. Where there were equivalent alternatives, I gave them weight 1/2 each. I then summed the number of occurrences of each node on a shortest path.

	A	B	C	D	E	F	G	H	I	J
A		-	B/E	E	-	-	E/F	F	FH	FHI
B			-	C/E	-	-	C/E/F	F	FH	FHI
C				-	-	B/E/G	-	G	GH	GHI
D					-	E/G	-	G	GH	GHI
E						-	-	F/G	F/GH	F/GHI
F							-	-	H	HI
G								-	H	HI
H									-	I
I										-
J										

Number of times each node occurs on a shortest path:

A	B	C	D	E	F	G	H	I	J
0	5/6	5/6	0	3 2/3	8 1/3	8 1/3	14	9	0

H occurs on more shortest paths between pairs of nodes than any of the other nodes.

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